260. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

The necessary and sufficient condition that a binary form be a perfect nth power is that its Hessian vanish.

I. Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Denoting $\frac{\partial u}{\partial x}$ by p, $\frac{\partial u}{\partial y}$ by q, the vanishing of the Hessian shows that $p=f(q)$, *i. e.*, $q=mp$, since both p and q are homogeneous and of the same degree. By Lagrange's method of solving partial differential equations, we have

$$\frac{dx}{m}=\frac{dy}{-1}=\frac{du}{0}.$$

Hence, u=constant, $x+my$=constant, and a general solution is given by

$$u=f(x+my)=(x+my)^n,$$

since u is homogeneous in x, y. It is easily verified that when $u=(x+my)^n$ the Hessian vanishes. Hence this condition is both necessary and sufficient.

II. Solution by the PROPOSER.

A slightly different point of view from the above is afforded by the following method:

The Hessian is the Jacobian of the first derivatives p and q. Hence $p-mq=0$. Also $xp+yq=nu$, n being the order of u. Solving for p and q,

$$p=\frac{nmu}{y+mx},\ q=\frac{nu}{y+mx}.$$

Also, $du=pdx+qdy=nu\dfrac{dy+mdx}{y+mx}$, or $\dfrac{du}{u}=n\dfrac{d(y+mx)}{y+mx}$.

Henge, $\log u=n\log k(y+mx)$, $u=(a_1x+a_2y)^n$.

261. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series, $\frac{1}{n^p}+\frac{3}{n^{2p}}+\frac{5}{n^{3p}}+\frac{7}{n^{4p}}+\frac{9}{n^{5p}}+\dots\dots$

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Denoting n^{-p} by x, we have

$$\sum_{i=1}^{\infty}\frac{(2i-1)}{n^{ip}}=x[1+3x+5x^2+7x^3+\dots\dots]$$
$$=x\sum(2r+1)x^r=2x\sum rx^r+x\sum x^r$$

$$=2x^2(1-x)^{-2}+x(1-x)^{-1}=x(1+x)(1-x)^{-2}=\frac{n^p+1}{(n^p-1)^2}$$

where we must have $|x|<1$.

Also solved by Henry Heaton, A. H. Holmes, and G. B. M. Zerr.

CALCULUS.

217. Proposed by Professor F. ANDEREGG, Oberlin College, Oberlin, Ohio.

Find $\lim_{n\doteq\infty}\frac{1}{n}\sqrt[n]{[(n+1)(n+2)\ldots\ldots(2n)]}$.

I. Solution by the PROPOSER.

Let $x=\lim_{n\doteq\infty}\frac{1}{n}\sqrt[n]{[(n+1)(n+2)\ldots\ldots 2n]}$

$$=\lim_{n\doteq\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)}.$$

Then $\log x=\lim_{n\doteq\infty}\frac{1}{n}\log\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)\right]$

$$=\lim_{n\doteq\infty}\frac{1}{n}\sum_{\lambda=1}^{\lambda=n}\log\left(1+\frac{\lambda}{n}\right)=\lim_{n\doteq\infty}\frac{1}{n}\sum_{\lambda=1}^{\lambda=n}\left(\frac{\lambda}{n}-\frac{\lambda^2}{2n^2}+\frac{\lambda^3}{3n^3}-\ldots\ldots\right)$$

$$=\lim_{n\doteq\infty}\sum_{\lambda=1}^{\lambda=n}\sum_{\kappa=1}^{\kappa=n}(-1)^{\kappa-1}\frac{\lambda^\kappa}{\kappa n^{\kappa+1}}.$$

If the method of differences is used for $\sum_{\lambda=1}^{\lambda=n}\lambda^\kappa=1^\kappa+2^\kappa+3^\kappa+\ldots\ldots$, the κth series of differences is

$$(\kappa+1)^\kappa-\binom{\kappa}{1}\kappa^\kappa+\binom{\kappa}{2}(\kappa-1)^\kappa-\binom{\kappa}{3}(\kappa-2)^\kappa+\ldots$$
$$+(-1)^{\kappa-1}\binom{\kappa}{\kappa-1}2^\kappa+(-1)^\kappa 1^\kappa\equiv\kappa!.$$

The $(\kappa+1)$th series is

$$(\kappa+2)^\kappa-\binom{\kappa+1}{1}(\kappa+1)^\kappa+\binom{\kappa+1}{2}\kappa^\kappa-\ldots+(-1)^\kappa\binom{\kappa+1}{\kappa}2^\kappa+(-1)^{\kappa+1}1^\kappa\equiv 0,$$

κ being a positive integer.

If the first given number is represented by a and the successive differences by d_1, d_2, ……

$$S_{n,\kappa}=\binom{n}{1}a+\binom{n}{2}d_1+\binom{n}{3}d_2+\ldots+\binom{n}{\kappa+1}d_\kappa.$$